Symbolic disintegration with a variety of base measures

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Chung-chieh Shan

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Exact calculation

for approximate computation

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for approximate computation

Automatic differentiation
for gradient descent

Automatic disintegration
for inference and sampling

Exact calculation
for approximate computation

Automatic differentiation
for gradient descent

Automatic disintegration
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A variety of base measures

$$(t,p) \leftsquigarrow \mu \quad = \quad \begin{array}{l} t \leftsquigarrow \beta \\ p \leftsquigarrow \kappa(t) \end{array} \quad : \mathbb{M}\,(T \times P)$$

Disintegration relates a joint measure $\mu : \mathbb{M}(T \times P)$

$$(t,p) \leftsquigarrow \mu \quad = \quad \begin{array}{l} t \leftsquigarrow \beta \\ p \leftsquigarrow \kappa(t) \end{array} \quad : \mathbb{M}(T \times P)$$

Disintegration relates a joint measure $\mu : \mathbb{M}\,(T \times P)$ with a base measure $\beta : \mathbb{M}\,T$

$$
\begin{aligned}
(t,p) \leftsquigarrow \mu \quad = \quad & t \leftsquigarrow \beta \\
& p \leftsquigarrow \kappa(t)
\end{aligned}
\qquad : \mathbb{M}\,(T \times P)
$$

Disintegration relates a joint measure $\mu : \mathbb{M}\,(T \times P)$ with a base measure $\beta : \mathbb{M}\,T$ and a kernel $\kappa : T \to \mathbb{M}\,P$

by the equation

$$(t, p) \leftsquigarrow \mu \quad = \quad \begin{array}{l} t \leftsquigarrow \beta \\ p \leftsquigarrow \kappa(t) \end{array} \quad : \mathbb{M}\,(T \times P)$$

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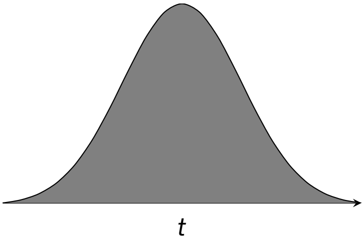
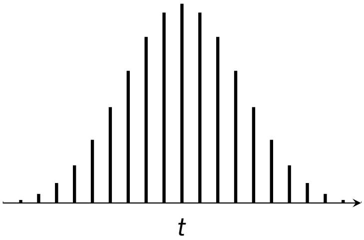
Generalizes density:

$$\begin{array}{l} t \leftsquigarrow \textbf{normal} \\ p = () \end{array} \quad = \quad \begin{array}{l} t \leftsquigarrow \textbf{lebesgue} \\ p \leftsquigarrow \textbf{factor}\ (\textbf{dnorm}\ t) \end{array} \quad : \mathbb{M}\,(\mathbb{R} \times \mathbb{1})$$





Disintegration relates a joint measure $\mu : \mathbb{M}(T \times P)$ with a base measure $\beta : \mathbb{M}\,T$ and a kernel $\kappa : T \to \mathbb{M}\,P$

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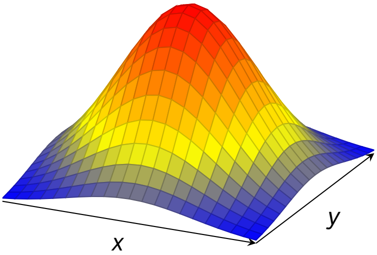
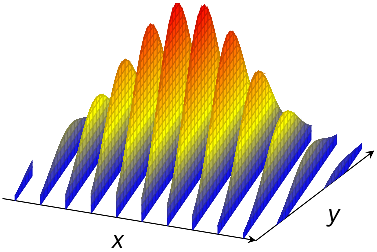
$$(t, p) \leftarrow \mu \quad = \quad \begin{array}{l} t \leftarrow \beta \\ p \leftarrow \kappa(t) \end{array} \quad : \mathbb{M}(T \times P)$$

Generalizes conditioning:

$$\begin{array}{l} x \leftarrow \textbf{normal} \\ y \leftarrow \textbf{normal} \\ t \leftarrow 5 \cdot x + 0.1 \cdot y \\ p = (x, y) \end{array} \quad = \quad \begin{array}{l} t \leftarrow \textbf{lebesgue} \\ p \leftarrow \ldots \Pr(p, t) \ldots \end{array} \quad : \mathbb{M}(\mathbb{R} \times \mathbb{R}^2)$$

"unnormalized conditioning"





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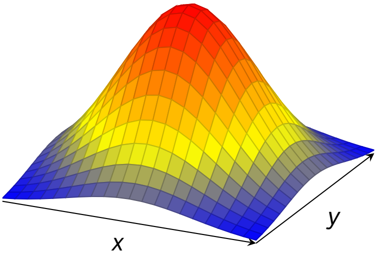

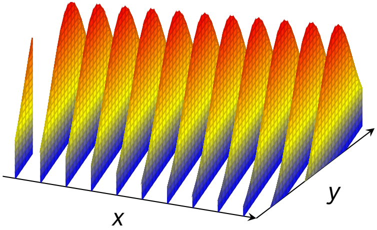

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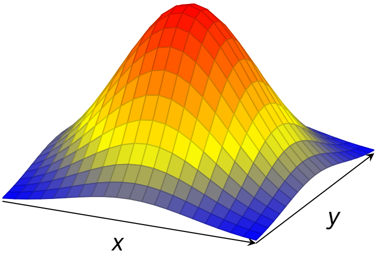

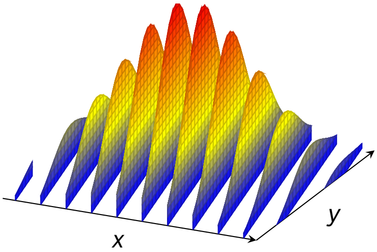

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Generalizes density and conditioning

Can be thought of as unnormalized conditioning

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A semantics-preserving transformation on probabilistic programs

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random choice (**normal**), scoring (**factor**)

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- s-finite measures/kernels
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- s-finite measures/kernels
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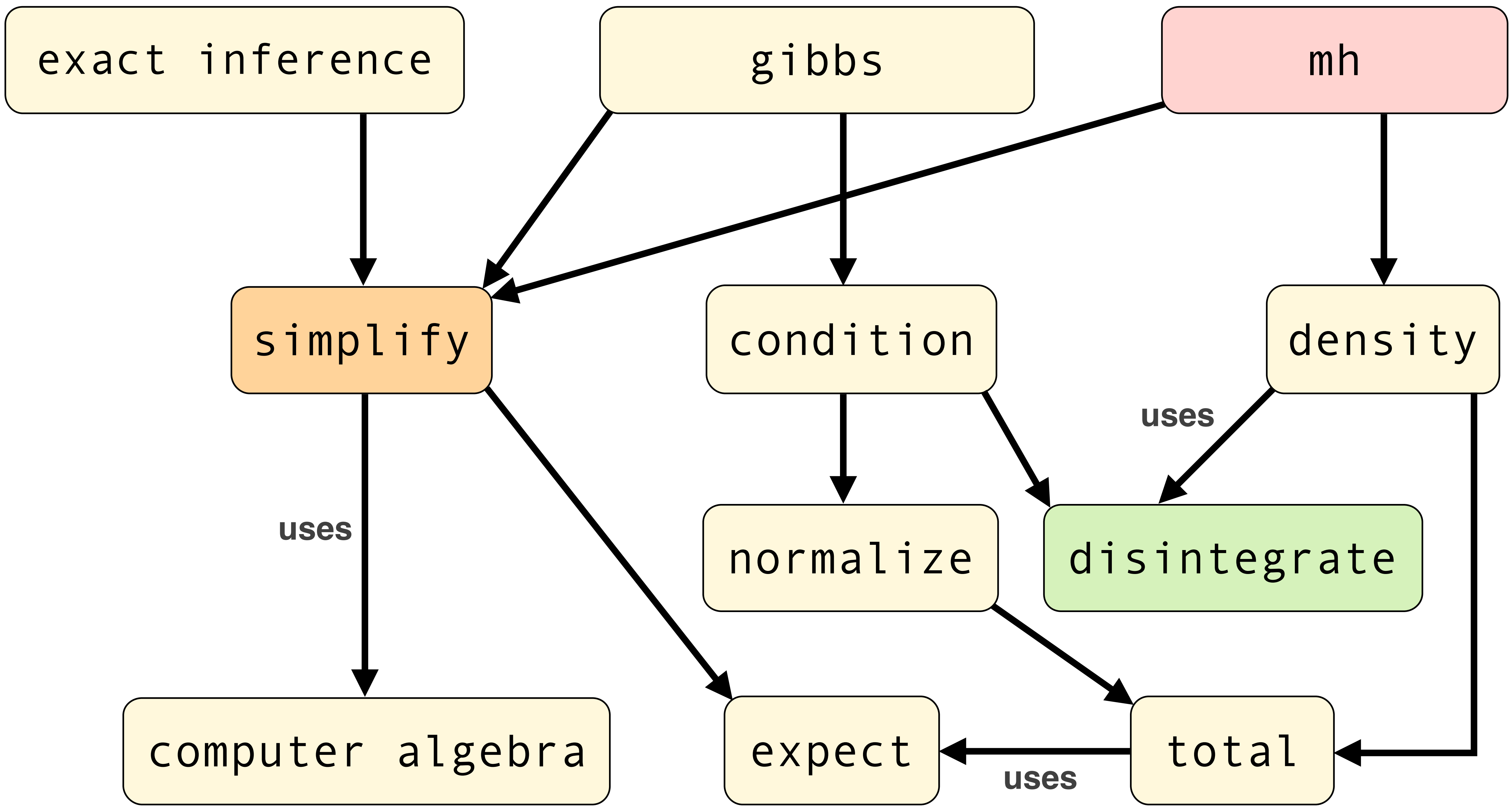
Derived by equational reasoning (hence proven sound by construction)

= This talk: equational reasoning on example programs

\+ Established PL technology: lazy and partial evaluation (traversing computation graph)

Composable transformations

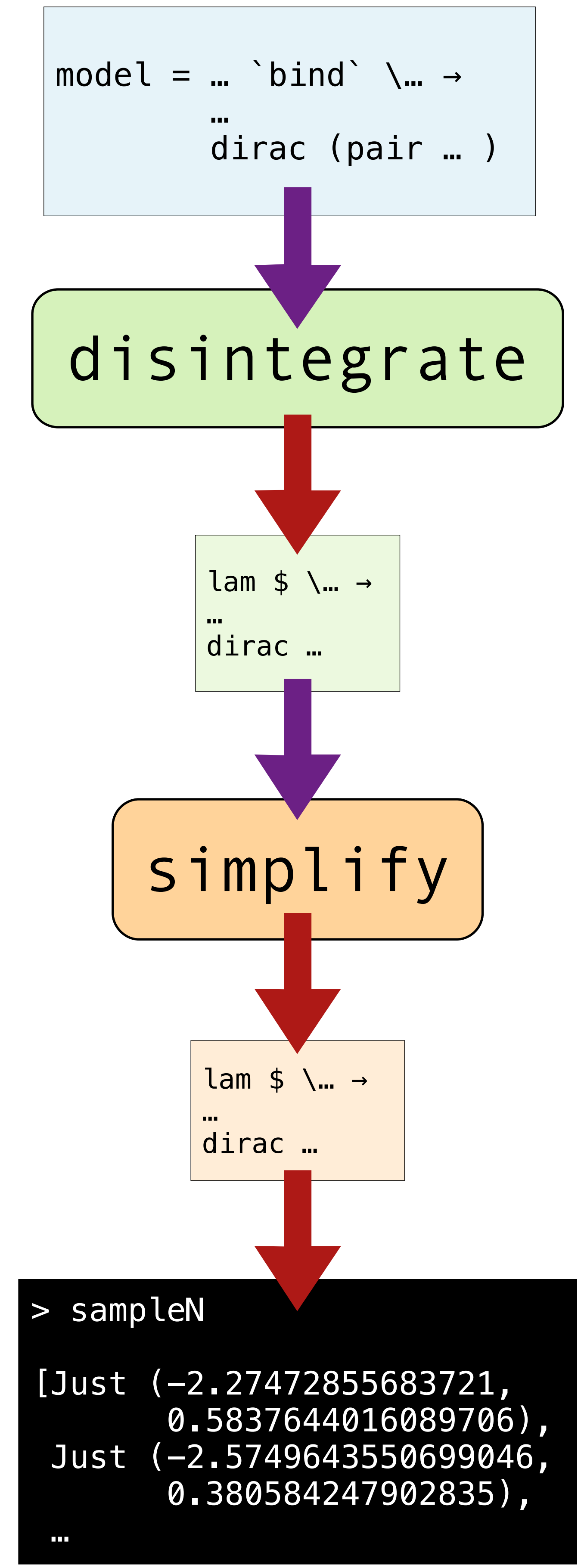


Modeling

Conditioning

Optimizing

Sampling



Optimizing

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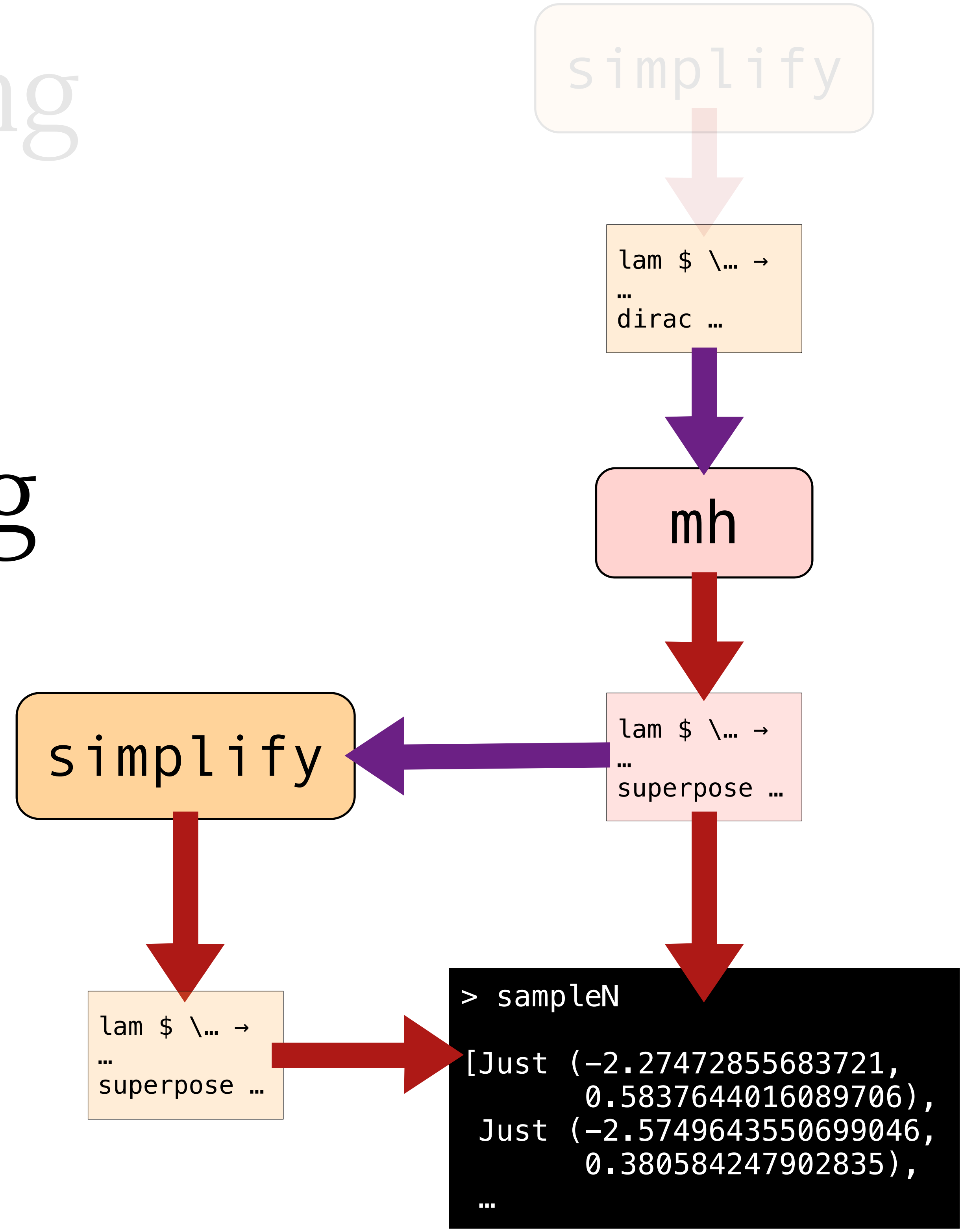

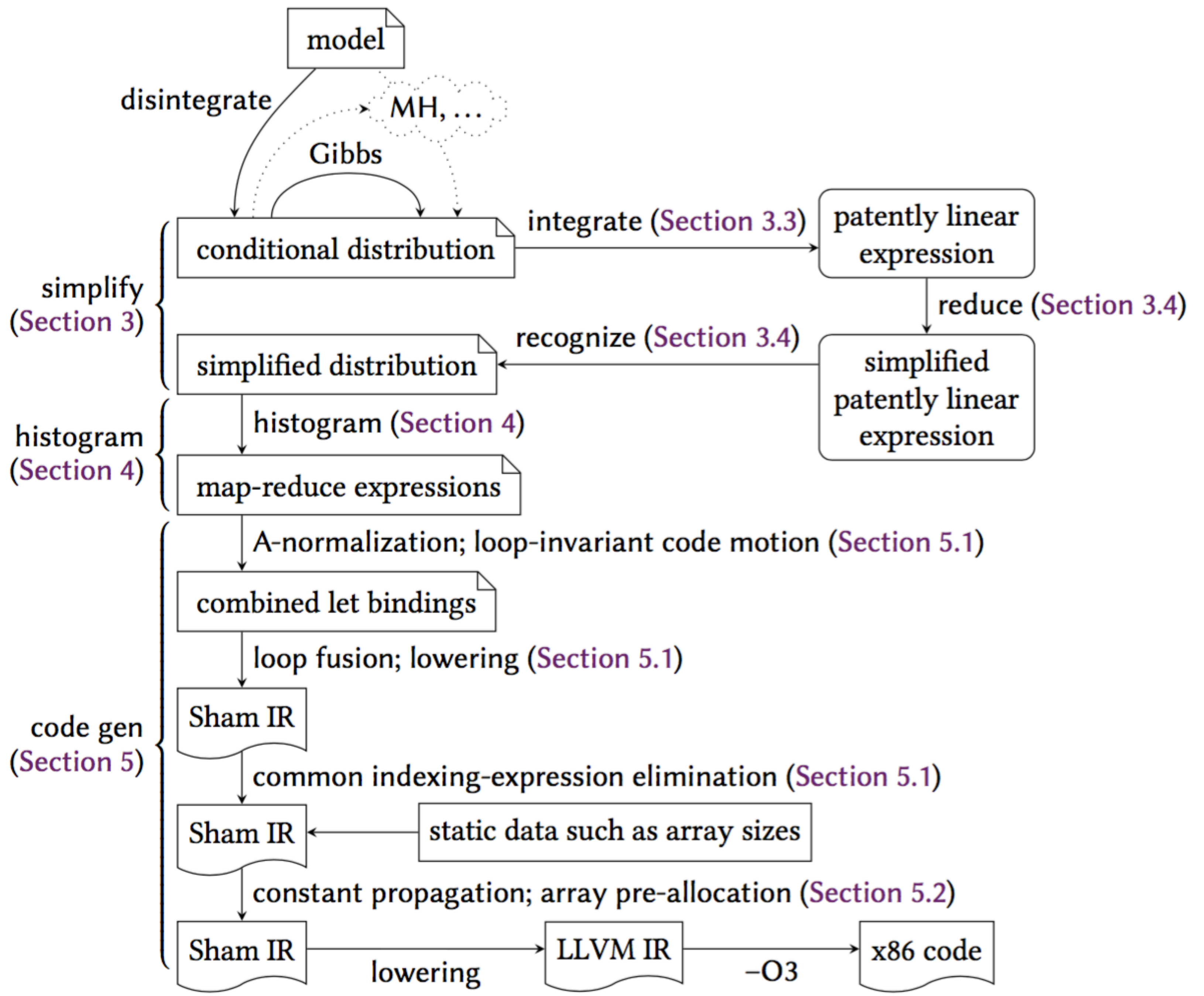

Fig. 1. Our pipeline, compiling probabilistic programs via math into imperative code to process data

A growing variety of base measures

Bhat et al. POPL 2012,
Shan & Ramsey POPL 2017

$$\beta ::= \textbf{lebesgue} \mid \textbf{counting} \mid \beta \otimes \beta$$

A growing variety of base measures

$$\beta ::= \overbrace{\beta \otimes \cdots \otimes \beta}^{n}$$

Narayanan & Shan
ICFP 2017

Bhat et al. POPL 2012,
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$$\beta ::= \textbf{lebesgue} \mid \textbf{counting} \mid \beta \otimes \beta$$

Narayanan & Shan

$$\beta ::= \textbf{ret}\ e \obar \beta \mid \beta \oplus \beta \mid (x \leftsquigarrow \beta) \otimes \beta$$

Discrete-continuous mixtures
Disjoint sums
Dependent products

A growing variety of base measures

$$\beta ::= \overbrace{\beta \otimes \cdots \otimes \beta}^{n}$$

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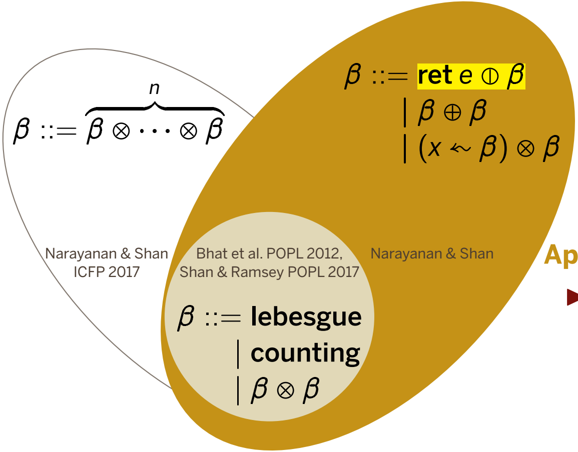
$$\beta ::= \textbf{ret}\ e \oplus \beta \mid \beta \oplus \beta \mid (x \leftsquigarrow \beta) \otimes \beta$$

Discrete-continuous mixtures
Disjoint sums
Dependent products

Applications:

- clamped model/observation
  (GPA, Tobit, camera)
  - likelihood ratio, importance sampling, mutual information
  - belief update, Gibbs sampling
- Metropolis-Hastings sampling
  - single site
  - reversible jump, light transport

A growing variety of base measures



Discrete-continuous mixtures
Disjoint sums
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To appear in TOPLAS

Applications:

- clamped model/observation (GPA, Tobit, camera)
  - likelihood ratio, importance sampling, mutual information
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  - single site
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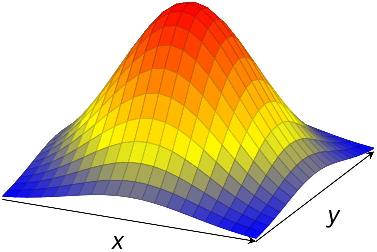
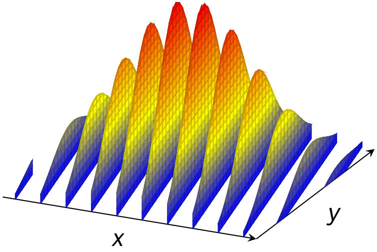
Fixed-base disintegration

$x \leftsquigarrow$ **normal**
$y \leftsquigarrow$ **normal**
$t = 5 \cdot x + 0.1 \cdot y$
$p = (x, y)$

$=$

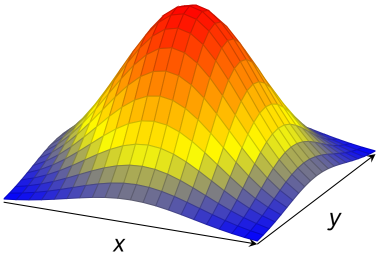
$t \leftsquigarrow$ **lebesgue**
$p \leftsquigarrow$ **???**

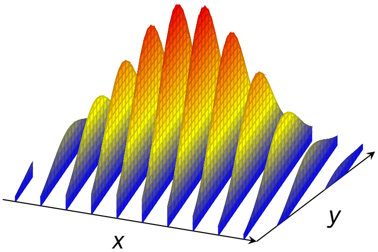




Fixed-base disintegration

$x \leftsquigarrow$ **normal** $y \leftsquigarrow$ **normal** $t = 5 \cdot x + 0.1 \cdot y$ $p = (x, y)$	$=$	$x \leftsquigarrow$ **normal** $y \leftsquigarrow$ **lebesgue** $t = 5 \cdot x + 0.1 \cdot y$ **factor** (**dnorm** y) $p = (x, y)$	$=$	$t \leftsquigarrow$ **lebesgue** $x \leftsquigarrow$ **normal** $y = 10 \cdot (t - 5 \cdot x)$ **factor** 10 **factor** (**dnorm** y) $p = (x, y)$





Fixed-base disintegration

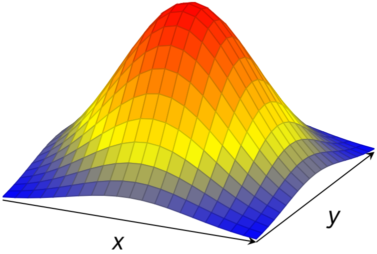
$x \leftsquigarrow$ **normal**
$y \leftsquigarrow$ **normal**
$t = 5 \cdot x + 0.1 \cdot y$
$p = (x, y)$

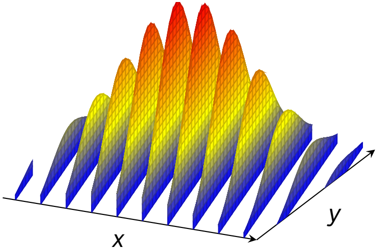
density

$x \leftsquigarrow$ **normal**
$y \leftsquigarrow$ **lebesgue**
$t = 5 \cdot x + 0.1 \cdot y$
factor (**dnorm** y)
$p = (x, y)$

reparameterize

$t \leftsquigarrow$ **lebesgue**
$x \leftsquigarrow$ **normal**
$y = 10 \cdot (t - 5 \cdot x)$
factor 10
factor (**dnorm** y)
$p = (x, y)$





Disintegration for *two* measurements

```
do {a ⇜ normal μ_a σ_a;
    b ⇜ normal μ_b σ_b;
    y_1 ⇜ normal (a · n_1 + b) 1;
    y_2 ⇜ normal (a · n_2 + b) 1;
    return ((y_1 , y_2) , (a , b))}
```

⇓

```
λ (t_1 , t_2) →
do {◁ (ȳ_1 , ȳ_2) (t_1 , t_2) [ā ⇜ normal μ_a σ_a;          ];
                              [b̄ ⇜ normal μ_b σ_b;          ]
                              [ȳ_1 ⇜ normal (a · n_1 + b) 1;]
                              [ȳ_2 ⇜ normal (a · n_2 + b) 1 ]
    return (a , b)}
```

⇓

```
λ (t_1 , t_2) →
do {◁ ȳ_1 t_1 [ā ⇜ normal μ_a σ_a;          ];
              [b̄ ⇜ normal μ_b σ_b;          ]
              [ȳ_1 ⇜ normal (a · n_1 + b) 1;]
              [ȳ_2 ⇜ normal (a · n_2 + b) 1 ]
    ◁ ȳ_2 t_2 ...;
    return (a , b)}
```

⇓

```
λ (t_1 , t_2) →
do {◁ ȳ_2 t_2 [ā ⇜ normal μ_a σ_a;                ];
              [b̄ ⇜ normal μ_b σ_b;                ]
              [factor (dNorm (a · n_1 + b) 1 t_1);]
              [ȳ_2 ⇜ normal (a · n_2 + b) 1       ]
    return (a , b)}
```

⇓

```
λ (t_1 , t_2) →
do {[ā ⇜ normal μ_a σ_a;                ];
    [b̄ ⇜ normal μ_b σ_b;                ]
    [factor (dNorm (a · n_1 + b) 1 t_1);]
    [factor (dNorm (a · n_2 + b) 1 t_2) ]
    return (a , b)}
```

⇓

```
λ (t_1 , t_2) →
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    b ⇜ normal μ_b σ_b;
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    factor (dNorm (a · n_2 + b) 1 t_2);
    return (a , b)}
```
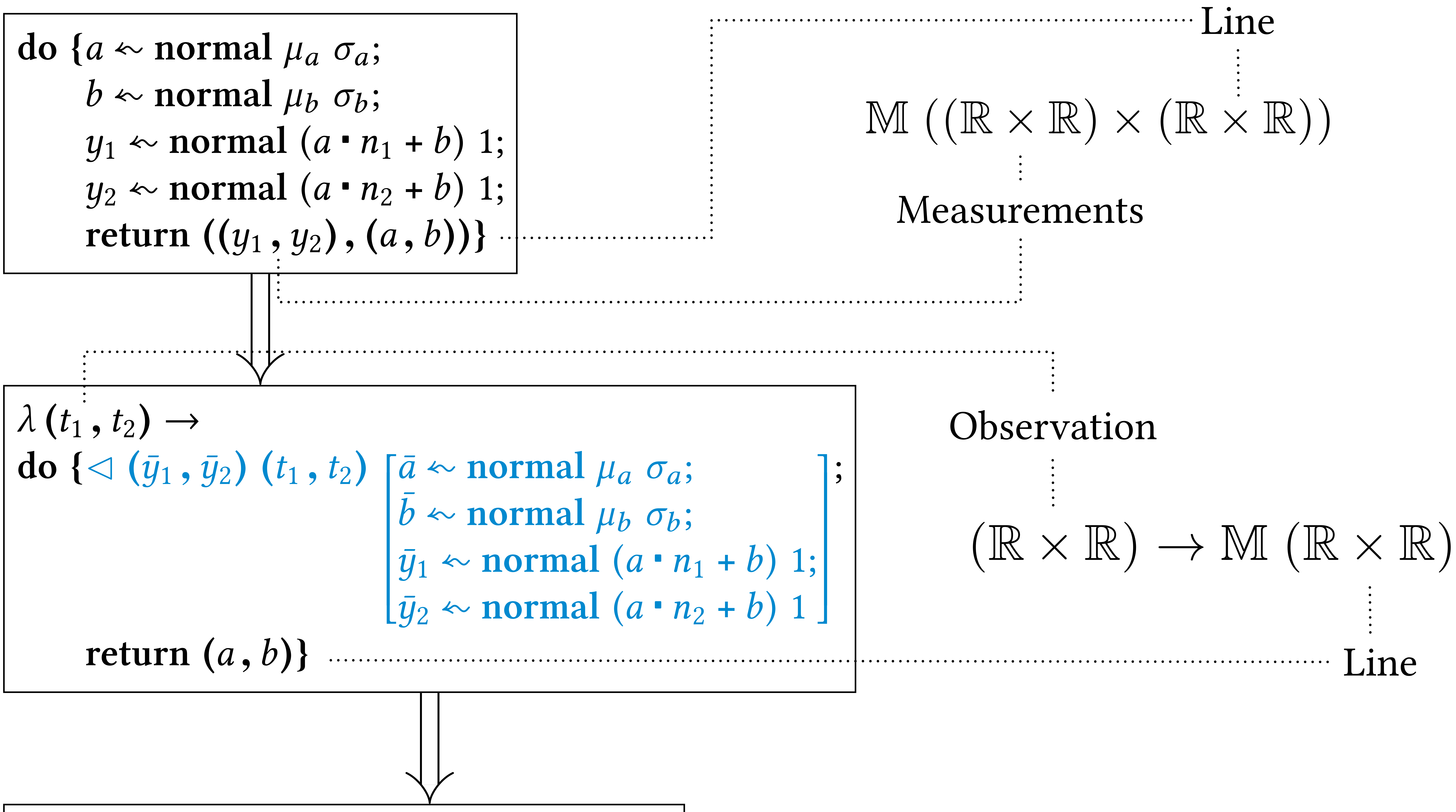


do {a ⇜ normal μa σa;
b ⇜ normal μb σb;
y1 ⇜ normal (a · n1 + b) 1;
y2 ⇜ normal (a · n2 + b) 1;
return ((y1 , y2) , (a , b))}
Line
M ((ℝ × ℝ) × (ℝ × ℝ))
Measurements
λ (t1 , t2) →
do {◁ (ȳ1 , ȳ2) (t1 , t2) [ā ⇜ normal μa σa;
b̄ ⇜ normal μb σb;
ȳ1 ⇜ normal (a · n1 + b) 1;
ȳ2 ⇜ normal (a · n2 + b) 1];
return (a , b)}
Observation
(ℝ × ℝ) → M (ℝ × ℝ)
Line

$$
\begin{aligned}
&\mathbf{do}\ \{a \leftsquigarrow \mathbf{normal}\ \mu_a\ \sigma_a;\\
&\quad b \leftsquigarrow \mathbf{normal}\ \mu_b\ \sigma_b;\\
&\quad y_1 \leftsquigarrow \mathbf{normal}\ (a \cdot n_1 + b)\ 1;\\
&\quad y_2 \leftsquigarrow \mathbf{normal}\ (a \cdot n_2 + b)\ 1;\\
&\quad \mathbf{return}\ ((y_1\,, y_2)\,, (a\,, b))\}
\end{aligned}
$$

Constrain the measurements

to be the observation

$$
\begin{aligned}
&\lambda\,(t_1\,, t_2) \to\\
&\mathbf{do}\ \{\lhd\ (\bar{y}_1\,, \bar{y}_2)\ (t_1\,, t_2)\ \begin{bmatrix} \bar{a} \leftsquigarrow \mathbf{normal}\ \mu_a\ \sigma_a;\\ \bar{b} \leftsquigarrow \mathbf{normal}\ \mu_b\ \sigma_b;\\ \bar{y}_1 \leftsquigarrow \mathbf{normal}\ (a \cdot n_1 + b)\ 1;\\ \bar{y}_2 \leftsquigarrow \mathbf{normal}\ (a \cdot n_2 + b)\ 1 \end{bmatrix};\\
&\quad \mathbf{return}\ (a\,, b)\}
\end{aligned}
$$

in the context of the ***heap***

$\lambda\,(t_1\,,\,t_2) \rightarrow$

$\textbf{do}\ \{\triangleleft\ (\bar{y}_1\,,\,\bar{y}_2)\ (t_1\,,\,t_2)\ \begin{bmatrix} \bar{a} \leftsquigarrow \textbf{normal}\ \mu_a\ \sigma_a; \\ \bar{b} \leftsquigarrow \textbf{normal}\ \mu_b\ \sigma_b; \\ \bar{y}_1 \leftsquigarrow \textbf{normal}\ (a \cdot n_1 + b)\ 1; \\ \bar{y}_2 \leftsquigarrow \textbf{normal}\ (a \cdot n_2 + b)\ 1 \end{bmatrix};$

$\quad\textbf{return}\ (a\,,\,b)\}$

$\lambda\,(t_1\,,\,t_2) \rightarrow$

$\textbf{do}\ \{\triangleleft\ \bar{y}_1\ t_1\ \begin{bmatrix} \bar{a} \leftsquigarrow \textbf{normal}\ \mu_a\ \sigma_a; \\ \bar{b} \leftsquigarrow \textbf{normal}\ \mu_b\ \sigma_b; \\ \bar{y}_1 \leftsquigarrow \textbf{normal}\ (a \cdot n_1 + b)\ 1; \\ \bar{y}_2 \leftsquigarrow \textbf{normal}\ (a \cdot n_2 + b)\ 1 \end{bmatrix};$

$\quad\triangleleft\ \bar{y}_2\ t_2\ \ldots;$

$\quad\textbf{return}\ (a\,,\,b)\}$

Constrain is defined for each construct

$\triangleleft\ e\ t\ h =$

case e of

$(e + e') \rightarrow \ \ldots$

$(e\,,\,e') \rightarrow \ \ldots$

$\ldots$

Two recursive calls to ***constrain***

```
λ (t₁ , t₂) →
do {◁ ȳ₁ t₁ [ā ⇜ normal μ_a σ_a;
             b̄ ⇜ normal μ_b σ_b;
             ȳ₁ ⇜ normal (a · n₁ + b) 1;
             ȳ₂ ⇜ normal (a · n₂ + b) 1 ];
    ◁ ȳ₂ t₂ ...;
    return (a , b)}
```

Replace binding with normal density

```
λ (t₁ , t₂) →
do {◁ ȳ₂ t₂ [ā ⇜ normal μ_a σ_a;
             b̄ ⇜ normal μ_b σ_b;
             factor (dNorm (a · n₁ + b) 1 t₁);
             ȳ₂ ⇜ normal (a · n₂ + b) 1 ];
    return (a , b)}
```

$$
\begin{array}{l}
\lambda\,(t_1\,,\,t_2) \rightarrow \\
\textbf{do}\ \{\lhd\ \bar{y}_2\ t_2\ \left[\begin{array}{l}
\bar{a} \leftsquigarrow \textbf{normal}\ \mu_a\ \sigma_a; \\
\bar{b} \leftsquigarrow \textbf{normal}\ \mu_b\ \sigma_b; \\
\textbf{factor}\ (\textbf{dNorm}\ (a \cdot n_1 + b)\ 1\ t_1); \\
\bar{y}_2 \leftsquigarrow \textbf{normal}\ (a \cdot n_2 + b)\ 1
\end{array}\right]; \\
\quad\quad \textbf{return}\ (a\,,\,b)\}
\end{array}
$$

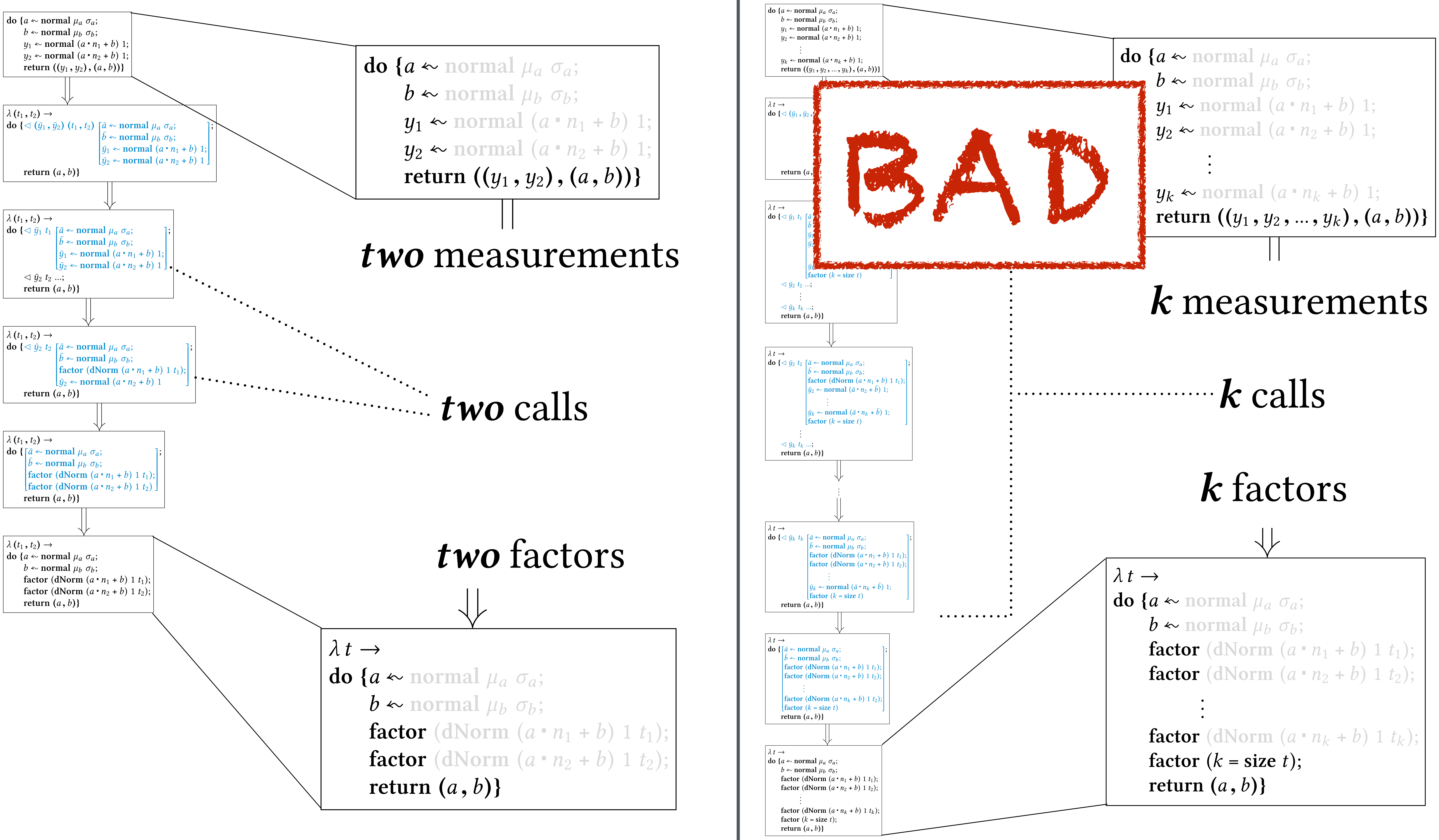
$$
\begin{array}{l}
\lambda\,(t_1\,,\,t_2) \rightarrow \\
\textbf{do}\ \{\left[\begin{array}{l}
\bar{a} \leftsquigarrow \textbf{normal}\ \mu_a\ \sigma_a; \\
\bar{b} \leftsquigarrow \textbf{normal}\ \mu_b\ \sigma_b; \\
\textbf{factor}\ (\textbf{dNorm}\ (a \cdot n_1 + b)\ 1\ t_1); \\
\textbf{factor}\ (\textbf{dNorm}\ (a \cdot n_2 + b)\ 1\ t_2)
\end{array}\right]; \\
\quad\quad \textbf{return}\ (a\,,\,b)\}
\end{array}
$$

Replace binding with normal density

$$
\begin{aligned}
&\lambda\,(t_1\,,\,t_2) \rightarrow\\
&\textbf{do}\ \{\left[\begin{array}{l}\bar{a} \leftsquigarrow \textbf{normal}\ \mu_a\ \sigma_a;\\ \bar{b} \leftsquigarrow \textbf{normal}\ \mu_b\ \sigma_b;\\ \textbf{factor}\ (\textbf{dNorm}\ (a \cdot n_1 + b)\ 1\ t_1);\\ \textbf{factor}\ (\textbf{dNorm}\ (a \cdot n_2 + b)\ 1\ t_2)\end{array}\right];\\
&\qquad \textbf{return}\ (a\,,\,b)\}
\end{aligned}
$$

Absorb heap
into
final output

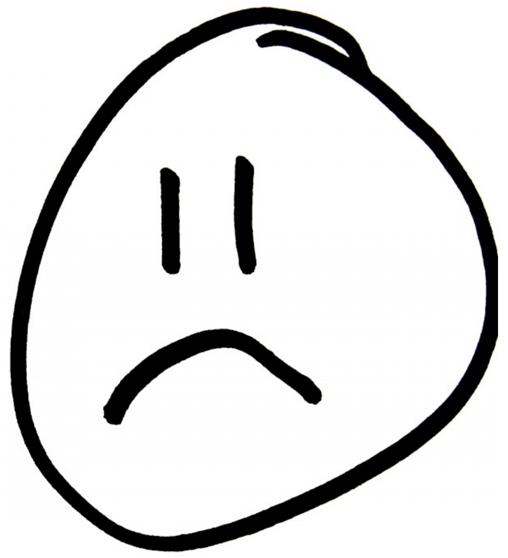
$$
\begin{aligned}
&\lambda\,(t_1\,,\,t_2) \rightarrow\\
&\textbf{do}\ \{a \leftsquigarrow \textbf{normal}\ \mu_a\ \sigma_a;\\
&\qquad b \leftsquigarrow \textbf{normal}\ \mu_b\ \sigma_b;\\
&\qquad \textbf{factor}\ (\textbf{dNorm}\ (a \cdot n_1 + b)\ 1\ t_1);\\
&\qquad \textbf{factor}\ (\textbf{dNorm}\ (a \cdot n_2 + b)\ 1\ t_2);\\
&\qquad \textbf{return}\ (a\,,\,b)\}
\end{aligned}
$$


do {a ↜ normal μa σa;
b ↜ normal μb σb;
y1 ↜ normal (a • n1 + b) 1;
y2 ↜ normal (a • n2 + b) 1;
return ((y1 , y2) , (a , b))}
two measurements
two calls
two factors
λ t →
do {a ↜ normal μa σa;
b ↜ normal μb σb;
factor (dNorm (a • n1 + b) 1 t1);
factor (dNorm (a • n2 + b) 1 t2);
return (a , b)}
BAD
do {a ↜ normal μa σa;
b ↜ normal μb σb;
y1 ↜ normal (a • n1 + b) 1;
y2 ↜ normal (a • n2 + b) 1;
yk ↜ normal (a • nk + b) 1;
return ((y1 , y2 , … , yk) , (a , b))}
k measurements
k calls
k factors
λ t →
do {a ↜ normal μa σa;
b ↜ normal μb σb;
factor (dNorm (a • n1 + b) 1 t1);
factor (dNorm (a • n2 + b) 1 t2);
factor (dNorm (a • nk + b) 1 tk);
factor (k = size t);
return (a , b)}

Unrolled

k calls

k factors



```
do {a ⇜ normal μ_a σ_a;
    b ⇜ normal μ_b σ_b;
    y_1 ⇜ normal (a • n_1 + b) 1;
    y_2 ⇜ normal (a • n_2 + b) 1;
        ⋮
    y_k ⇜ normal (a • n_k + b) 1;
    return ((y_1 , y_2 , … , y_k) , (a , b))}
```

```
λ t →
do {◁ (ȳ_1 , ȳ_2 , … , ȳ_k) t [ā ⇜ normal μ_a σ_a;
                               b̄ ⇜ normal μ_b σ_b;
                               ȳ_1 ⇜ normal (ā • n_1 + b̄) 1;
                               ȳ_2 ⇜ normal (ā • n_2 + b̄) 1;
                                   ⋮
                               ȳ_k ⇜ normal (ā • n_k + b̄) 1];
    return (a , b)}
```

```
λ t →
do {◁ ȳ_1 t_1 [ā ⇜ normal μ_a σ_a;
               b̄ ⇜ normal μ_b σ_b;
               ȳ_1 ⇜ normal (ā • n_1 + b̄) 1;
               ȳ_2 ⇜ normal (ā • n_2 + b̄) 1;
                   ⋮
               ȳ_k ⇜ normal (ā • n_k + b̄) 1;
               factor (k = size t)];
    ◁ ȳ_2 t_2 …;
        ⋮
    ◁ ȳ_k t_k …;
    return (a , b)}
```

```
λ t →
do {◁ ȳ_2 t_2 [ā ⇜ normal μ_a σ_a;
               b̄ ⇜ normal μ_b σ_b;
               factor (dNorm (a • n_1 + b) 1 t_1);
               ȳ_2 ⇜ normal (ā • n_2 + b̄) 1;
                   ⋮
               ȳ_k ⇜ normal (ā • n_k + b̄) 1;
               factor (k = size t)];
        ⋮
    ◁ ȳ_k t_k …;
    return (a , b)}
```

⋮

```
λ t →
do {◁ ȳ_k t_k [ā ⇜ normal μ_a σ_a;
               b̄ ⇜ normal μ_b σ_b;
               factor (dNorm (a • n_1 + b) 1 t_1);
               factor (dNorm (a • n_2 + b) 1 t_2);
                   ⋮
               ȳ_k ⇜ normal (ā • n_k + b̄) 1;
               factor (k = size t)];
    return (a , b)}
```

```
λ t →
do {[ā ⇜ normal μ_a σ_a;
     b̄ ⇜ normal μ_b σ_b;
     factor (dNorm (a • n_1 + b) 1 t_1);
     factor (dNorm (a • n_2 + b) 1 t_2);
         ⋮
     factor (dNorm (a • n_k + b) 1 t_2);
     factor (k = size t)];
    return (a , b)}
```

```
λ t →
do {a ⇜ normal μ_a σ_a;
    b ⇜ normal μ_b σ_b;
    factor (dNorm (a • n_1 + b) 1 t_1);
    factor (dNorm (a • n_2 + b) 1 t_2);
        ⋮
    factor (dNorm (a • n_k + b) 1 t_k);
    factor (k = size t);
    return (a , b)}
```

```
do {a ⇜ normal μ_a σ_a;
    b ⇜ normal μ_b σ_b;
    y ⇜ plate k (λ i → normal (a • n_i + b) 1);
    return (y , (a , b))}
```

```
λ t →
do {◁ ȳ t [ā ⇜ normal μ_a σ_a;
           b̄ ⇜ normal μ_b σ_b;
           ȳ ⇜ plate k (λ i → normal (a • n_i + b) 1)];
    return (a , b)}
```

```
λ t →
do {◁ [ĵ·k] ȳ[j] t_ĵ [ā ⇜ []. normal μ_a σ_a;
                      b̄ ⇜ []. normal μ_b σ_b;
                      ȳ ⇜ [î·k]. normal (ā[] • n_î + b̄[]) 1;
                      factor []. (k = size t)];
    return (a , b)}
```

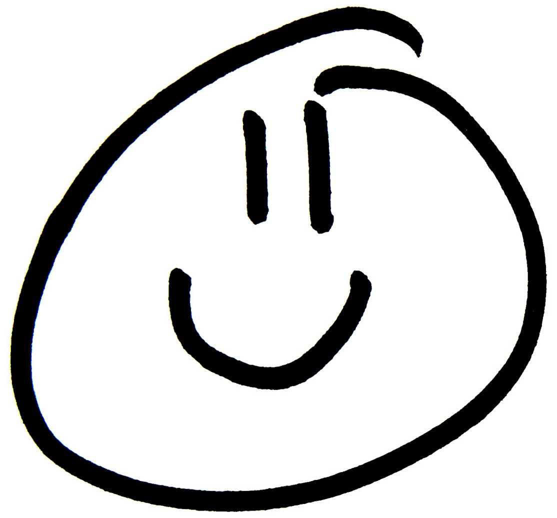
```
λ t →
do {[ā ⇜ []. normal μ_a σ_a;
     b̄ ⇜ []. normal μ_b σ_b;
     factor [ĵ·k]. (dNorm (ā[] • n_ĵ + b̄[]) 1 t_ĵ);
     factor []. (k = size t);
    return (a , b)}
```

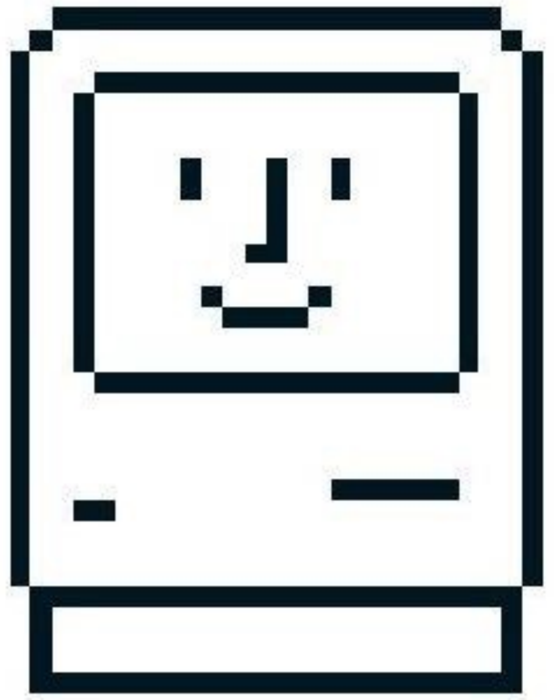
```
λ t →
do {a ⇜ normal μ_a σ_a;
    b ⇜ normal μ_b σ_b;
    _ ⇜ plate k (λ i → factor (dNorm (a • n_i + b) 1 t_i))
    factor (k = size t);
    return (a , b)}
```

Lifted

one call

one factor






Unrolled
Lifted
do {a ⇜ normal μa σa;
b ⇜ normal μb σb;
y1 ⇜ normal (a · n1 + b) 1;
y2 ⇜ normal (a · n2 + b) 1;
⋮
yk ⇜ normal (a · nk + b) 1;
return ((y1 , y2 , … , yk) , (a , b))}
become
do {a ⇜ normal μa σa;
b ⇜ normal μb σb;
y ⇜ plate k (λ i → normal (a · ni + b) 1);
return (y , (a , b))}
array
k-tuple
plate : ℕ → (ℕ → 𝕄 α) → 𝕄 ⟨α⟩
size
indexed dist.
dist. over arrays

Unrolled

$$
\begin{aligned}
&\textbf{do } \{a \leftsquigarrow \textbf{normal } \mu_a\ \sigma_a;\\
&\quad b \leftsquigarrow \textbf{normal } \mu_b\ \sigma_b;\\
&\quad y_1 \leftsquigarrow \textbf{normal } (a \cdot n_1 + b)\ 1;\\
&\quad y_2 \leftsquigarrow \textbf{normal } (a \cdot n_2 + b)\ 1;\\
&\qquad\qquad \vdots\\
&\quad y_k \leftsquigarrow \textbf{normal } (a \cdot n_k + b)\ 1;\\
&\quad \textbf{return } ((y_1, y_2, \ldots, y_k), (a, b))\}
\end{aligned}
$$

$$
\begin{aligned}
&\lambda\, t \rightarrow\\
&\textbf{do } \{\triangleleft\ (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_k)\ t\ \begin{bmatrix} \bar{a} \leftsquigarrow \textbf{normal } \mu_a\ \sigma_a;\\ \bar{b} \leftsquigarrow \textbf{normal } \mu_b\ \sigma_b;\\ \bar{y}_1 \leftsquigarrow \textbf{normal } (\bar{a} \cdot n_1 + \bar{b})\ 1;\\ \bar{y}_2 \leftsquigarrow \textbf{normal } (\bar{a} \cdot n_2 + \bar{b})\ 1;\\ \vdots\\ \bar{y}_k \leftsquigarrow \textbf{normal } (\bar{a} \cdot n_k + \bar{b})\ 1 \end{bmatrix};\\
&\quad \textbf{return } (a, b)\}
\end{aligned}
$$

Lifted

$$
\begin{aligned}
&\textbf{do } \{a \leftsquigarrow \textbf{normal } \mu_a\ \sigma_a;\\
&\quad b \leftsquigarrow \textbf{normal } \mu_b\ \sigma_b;\\
&\quad y \leftsquigarrow \textbf{plate } k\ (\lambda\, i \rightarrow \textbf{normal } (a \cdot n_i + b)\ 1);\\
&\quad \textbf{return } (y, (a, b))\}
\end{aligned}
$$

Unrolled

$$
\begin{aligned}
&\mathbf{do}\ \{a \leftsquigarrow \mathbf{normal}\ \mu_a\ \sigma_a;\\
&\quad b \leftsquigarrow \mathbf{normal}\ \mu_b\ \sigma_b;\\
&\quad y_1 \leftsquigarrow \mathbf{normal}\ (a \cdot n_1 + b)\ 1;\\
&\quad y_2 \leftsquigarrow \mathbf{normal}\ (a \cdot n_2 + b)\ 1;\\
&\qquad\quad \vdots\\
&\quad y_k \leftsquigarrow \mathbf{normal}\ (a \cdot n_k + b)\ 1;\\
&\quad \mathbf{return}\ ((y_1, y_2, \ldots, y_k), (a, b))\}
\end{aligned}
$$

$$\Downarrow$$

$$
\begin{aligned}
&\lambda\, t \rightarrow\\
&\mathbf{do}\ \{\triangleleft\ (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_k)\ t \begin{bmatrix} \bar{a} \leftsquigarrow \mathbf{normal}\ \mu_a\ \sigma_a;\\ \bar{b} \leftsquigarrow \mathbf{normal}\ \mu_b\ \sigma_b;\\ \bar{y}_1 \leftsquigarrow \mathbf{normal}\ (\bar{a} \cdot n_1 + \bar{b})\ 1;\\ \bar{y}_2 \leftsquigarrow \mathbf{normal}\ (\bar{a} \cdot n_2 + \bar{b})\ 1;\\ \vdots\\ \bar{y}_k \leftsquigarrow \mathbf{normal}\ (\bar{a} \cdot n_k + \bar{b})\ 1 \end{bmatrix};\\
&\quad \mathbf{return}\ (a, b)\}
\end{aligned}
$$

$$\Downarrow$$

$$\lambda\, t \rightarrow$$

Lifted

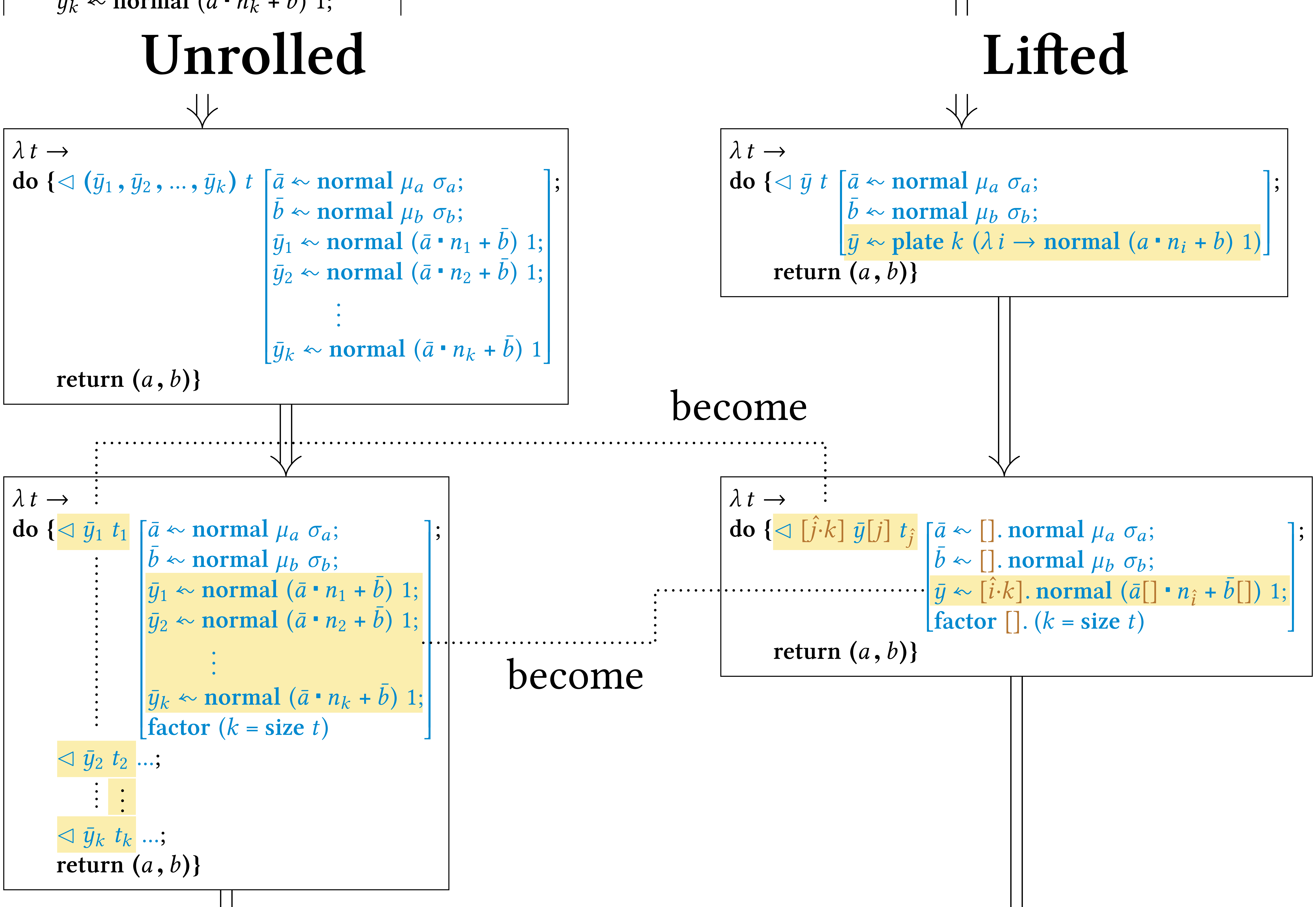
$$
\begin{aligned}
&\mathbf{do}\ \{a \leftsquigarrow \mathbf{normal}\ \mu_a\ \sigma_a;\\
&\quad b \leftsquigarrow \mathbf{normal}\ \mu_b\ \sigma_b;\\
&\quad y \leftsquigarrow \mathbf{plate}\ k\ (\lambda\, i \rightarrow \mathbf{normal}\ (a \cdot n_i + b)\ 1);\\
&\quad \mathbf{return}\ (y, (a, b))\}
\end{aligned}
$$

$$\Downarrow$$

$$
\begin{aligned}
&\lambda\, t \rightarrow\\
&\mathbf{do}\ \{\triangleleft\ \bar{y}\ t \begin{bmatrix} \bar{a} \leftsquigarrow \mathbf{normal}\ \mu_a\ \sigma_a;\\ \bar{b} \leftsquigarrow \mathbf{normal}\ \mu_b\ \sigma_b;\\ \bar{y} \leftsquigarrow \mathbf{plate}\ k\ (\lambda\, i \rightarrow \mathbf{normal}\ (a \cdot n_i + b)\ 1) \end{bmatrix};\\
&\quad \mathbf{return}\ (a, b)\}
\end{aligned}
$$

$$\Downarrow$$

$$\lambda\, t \rightarrow$$



Unrolled
Lifted
λ t →
do {◁ (ȳ1 , ȳ2 , … , ȳk) t [ā ⇜ normal μa σa;
b̄ ⇜ normal μb σb;
ȳ1 ⇜ normal (ā · n1 + b̄) 1;
ȳ2 ⇜ normal (ā · n2 + b̄) 1;
ȳk ⇜ normal (ā · nk + b̄) 1];
return (a , b)}
λ t →
do {◁ ȳ t [ā ⇜ normal μa σa;
b̄ ⇜ normal μb σb;
ȳ ⇜ plate k (λ i → normal (a · ni + b) 1)];
return (a , b)}
become
λ t →
do {◁ ȳ1 t1 [ā ⇜ normal μa σa;
b̄ ⇜ normal μb σb;
ȳ1 ⇜ normal (ā · n1 + b̄) 1;
ȳ2 ⇜ normal (ā · n2 + b̄) 1;
ȳk ⇜ normal (ā · nk + b̄) 1;
factor (k = size t)];
◁ ȳ2 t2 …;
◁ ȳk tk …;
return (a , b)}
become
λ t →
do {◁ [ĵ·k] ȳ[j] tĵ [ā ⇜ []. normal μa σa;
b̄ ⇜ []. normal μb σb;
ȳ ⇜ [î·k]. normal (ā[] · nî + b̄[]) 1;
factor []. (k = size t)];
return (a , b)}

Lifted

$$
\begin{array}{l}
\lambda\, t \to \\
\textbf{do}\ \{ \triangleleft\ [\hat{j}\cdot k]\ \bar{y}[j]\ t_{\hat{j}}\ \left[\begin{array}{l} \bar{a} \leftsquigarrow [].\ \textbf{normal}\ \mu_a\ \sigma_a; \\ \bar{b} \leftsquigarrow [].\ \textbf{normal}\ \mu_b\ \sigma_b; \\ \bar{y} \leftsquigarrow [\hat{i}\cdot k].\ \textbf{normal}\ (\bar{a}[] \cdot n_{\hat{i}} + \bar{b}[])\ 1; \\ \textbf{factor}\ [].\ (k = \textbf{size}\ t) \end{array} \right]; \\
\qquad \textbf{return}\ (a, b) \}
\end{array}
$$

Lifted constrain

Lifted binding

```
[ȳₖ ⇜ normal (ā · nₖ + b̄) 1]
```

Unrolled

```
λ t →
do {◁ ȳ₁ t₁ [ā ⇜ normal μₐ σₐ;
             b̄ ⇜ normal μ_b σ_b;
             ȳ₁ ⇜ normal (ā · n₁ + b̄) 1;
             ȳ₂ ⇜ normal (ā · n₂ + b̄) 1;
                      ⋮
             ȳₖ ⇜ normal (ā · nₖ + b̄) 1;
             factor (k = size t)                ];
    ◁ ȳ₂ t₂ ...;
          ⋮
    ◁ ȳₖ tₖ ...;
    return (a, b)}
```

```
λ t →
do {◁ ȳ₂ t₂ [ā ⇜ normal μₐ σₐ;
             b̄ ⇜ normal μ_b σ_b;
             factor (dNorm (a · n₁ + b) 1 t₁);
             ȳ₂ ⇜ normal (ā · n₂ + b̄) 1;
                      ⋮
             ȳₖ ⇜ normal (ā · nₖ + b̄) 1;
             factor (k = size t)                ];
          ⋮
    ◁ ȳₖ tₖ ...;
    return (a, b)}
```

Lifted

```
λ t →
do {◁ [ĵ·k] ȳ[j] t_ĵ [ā ⇜ []. normal μₐ σₐ;
                      b̄ ⇜ []. normal μ_b σ_b;
                      ȳ ⇜ [î·k]. normal (ā[] · n_î + b̄[]) 1;
                      factor []. (k = size t)                ];
    return (a, b)}
```

Unrolled

$\lambda\, t \rightarrow$
do {$\triangleleft\ \bar{y}_2\ t_2$ $\left[\begin{array}{l} \bar{a} \leftsquigarrow \textbf{normal}\ \mu_a\ \sigma_a; \\ \bar{b} \leftsquigarrow \textbf{normal}\ \mu_b\ \sigma_b; \\ \textbf{factor}\ (\textbf{dNorm}\ (a \cdot n_1 + b)\ 1\ t_1); \\ \bar{y}_2 \leftsquigarrow \textbf{normal}\ (\bar{a} \cdot n_2 + \bar{b})\ 1; \\ \quad\vdots \\ \bar{y}_k \leftsquigarrow \textbf{normal}\ (\bar{a} \cdot n_k + \bar{b})\ 1; \\ \textbf{factor}\ (k = \textbf{size}\ t) \end{array}\right]$;
$\vdots$
$\triangleleft\ \bar{y}_k\ t_k\ \ldots$;
return (a, b)}

$\vdots$

$\lambda\, t \rightarrow$
do {$\triangleleft\ \bar{y}_k\ t_k$ $\left[\begin{array}{l} \bar{a} \leftsquigarrow \textbf{normal}\ \mu_a\ \sigma_a; \\ \bar{b} \leftsquigarrow \textbf{normal}\ \mu_b\ \sigma_b; \\ \textbf{factor}\ (\textbf{dNorm}\ (a \cdot n_1 + b)\ 1\ t_1); \\ \textbf{factor}\ (\textbf{dNorm}\ (a \cdot n_2 + b)\ 1\ t_2); \\ \quad\vdots \\ \bar{y}_k \leftsquigarrow \textbf{normal}\ (\bar{a} \cdot n_k + \bar{b})\ 1; \\ \textbf{factor}\ (k = \textbf{size}\ t) \end{array}\right]$;
return (a, b)}

Lifted

Unrolled

$$\lambda\, t \to$$
$$\mathbf{do}\ \{\lhd\ \bar{y}_k\ t_k\ \left[\begin{array}{l} \bar{a} \leftsquigarrow \mathbf{normal}\ \mu_a\ \sigma_a; \\ \bar{b} \leftsquigarrow \mathbf{normal}\ \mu_b\ \sigma_b; \\ \text{factor}\ (\text{dNorm}\ (a \cdot n_1 + b)\ 1\ t_1); \\ \text{factor}\ (\text{dNorm}\ (a \cdot n_2 + b)\ 1\ t_2); \\ \quad\vdots \\ \bar{y}_k \leftsquigarrow \mathbf{normal}\ (\bar{a} \cdot n_k + \bar{b})\ 1; \\ \text{factor}\ (k = \text{size}\ t) \end{array}\right];$$
$$\mathbf{return}\ (a, b)\}$$

$$\lambda\, t \to$$
$$\mathbf{do}\ \{\left[\begin{array}{l} \bar{a} \leftsquigarrow \mathbf{normal}\ \mu_a\ \sigma_a; \\ \bar{b} \leftsquigarrow \mathbf{normal}\ \mu_b\ \sigma_b; \\ \text{factor}\ (\text{dNorm}\ (a \cdot n_1 + b)\ 1\ t_1); \\ \text{factor}\ (\text{dNorm}\ (a \cdot n_2 + b)\ 1\ t_2); \\ \quad\vdots \\ \text{factor}\ (\text{dNorm}\ (a \cdot n_k + b)\ 1\ t_2); \\ \text{factor}\ (k = \text{size}\ t) \end{array}\right];$$
$$\mathbf{return}\ (a, b)\}$$

become

Lifted

$$\lambda\, t \to$$
$$\mathbf{do}\ \{\left[\begin{array}{l} \bar{a} \leftsquigarrow [\,].\ \mathbf{normal}\ \mu_a\ \sigma_a; \\ \bar{b} \leftsquigarrow [\,].\ \mathbf{normal}\ \mu_b\ \sigma_b; \\ \text{factor}\ [\hat{j} \cdot k].\ (\text{dNorm}\ (\bar{a}[\,] \cdot n_{\hat{j}} + \bar{b}[\,])\ 1\ t_{\hat{j}}); \\ \text{factor}\ [\,].\ (k = \text{size}\ t); \end{array}\right];$$
$$\mathbf{return}\ (a, b)\}$$

[factor (k = size t)   ]

Unrolled

```
λ t →
do {[ā ↜ normal μ_a σ_a;
     b̄ ↜ normal μ_b σ_b;
     factor (dNorm (a · n_1 + b) 1 t_1);
     factor (dNorm (a · n_2 + b) 1 t_2);
              ⋮
     factor (dNorm (a · n_k + b) 1 t_2);
     factor (k = size t)                 ];
    return (a, b)}
```

```
λ t →
do {a ↜ normal μ_a σ_a;
    b ↜ normal μ_b σ_b;
    factor (dNorm (a · n_1 + b) 1 t_1);
    factor (dNorm (a · n_2 + b) 1 t_2);
             ⋮
    factor (dNorm (a · n_k + b) 1 t_k);
    factor (k = size t);
    return (a, b)}
```

Lifted

```
λ t →
do {[ā ↜ []. normal μ_a σ_a;
     b̄ ↜ []. normal μ_b σ_b;
     factor [ĵ·k]. (dNorm (ā[] · n_ĵ + b̄[]) 1 t_ĵ);
     factor []. (k = size t);                       ];
    return (a, b)}
```

```
λ t →
do {a ↜ normal μ_a σ_a;
    b ↜ normal μ_b σ_b;
    _ ↜ plate k (λ i → factor (dNorm (a · n_i + b) 1 t_i))
    factor (k = size t);
    return (a, b)}
```

$$naiveBayes \; : \mathbb{N} \to \mathbb{N} \to \mathbb{N} \to \mathbb{N} \to \mathbb{M}\,(\langle\langle\mathbb{N}\rangle\rangle \times \langle\mathbb{N}\rangle) \tag{6.38}$$

$$\begin{aligned} naiveBayes = \lambda sizeVocab.\, \lambda numLabels.\, \lambda numDocs.\, \lambda sizeEachDoc. \\ \mathbf{do}\ \{\beta \leftsquigarrow \mathbf{plate}\ numLabels\ \_.\ \mathbf{dirichlet}\ (\mathbf{array}\ sizeVocab\ \_.\ 1); \\ \theta \leftsquigarrow \mathbf{dirichlet}\ (\mathbf{array}\ numLabels\ \_.\ 1); \\ \zeta \leftsquigarrow \mathbf{plate}\ numDocs\ \_.\ \mathbf{categorical}\ \theta; \\ ds \leftsquigarrow \mathbf{plate}\ numDocs\ i. \\ \mathbf{plate}\ sizeEachDoc\ \_.\ \mathbf{categorical}\ \beta[\zeta[i]]; \\ \mathbf{return}\ (ds, \zeta)\} \end{aligned} \tag{6.39}$$

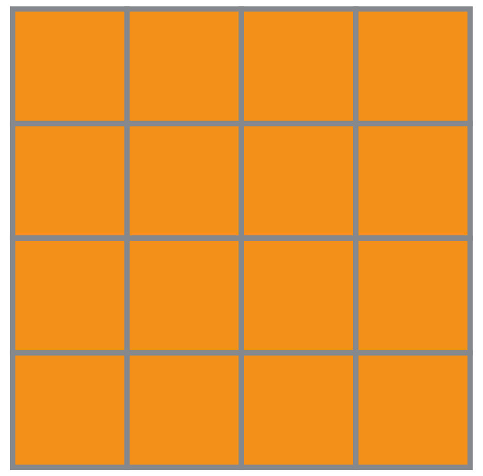
$$\mathit{disintegrate}^{\diamond}\ (\mathit{naiveBayes}\ \mathit{sizeVocab}\ \mathit{numLabels}\ \mathit{numDocs}\ \mathit{sizeEachDoc})\ t \tag{6.41}$$

$$\begin{aligned}
\Longrightarrow \mathbf{do}\ \{&\beta \leftsquigarrow \mathbf{plate}\ \mathit{numLabels}\ _.\ \mathbf{dirichlet}\ (\mathbf{array}\ \mathit{sizeVocab}\ _.\ 1); \\
&\theta \leftsquigarrow \mathbf{dirichlet}\ (\mathbf{array}\ \mathit{numLabels}\ _.\ 1); \\
&\zeta \leftsquigarrow \mathbf{plate}\ \mathit{numDocs}\ _.\ \mathbf{categorical}\ \theta; \\
&_ \leftsquigarrow \mathbf{plate}\ \mathit{numDocs}\ i. \\
&\qquad \mathbf{plate}\ \mathit{sizeEachDoc}\ j. \\
&\qquad\quad \mathbf{do}\ \{\mathbf{factor}\ (\beta[\zeta[i]][t[i][j]] \div \mathit{sum}\ \beta[\zeta[i]]); \\
&\qquad\qquad\quad \mathbf{return}\ ()\}; \\
&\mathbf{return}\ \zeta\}
\end{aligned} \tag{6.42}$$

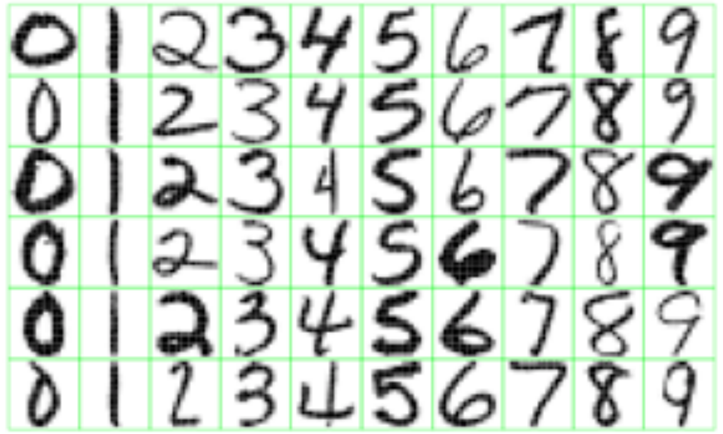
What works

$\hat{i} \cdot k$

Index variable — Size

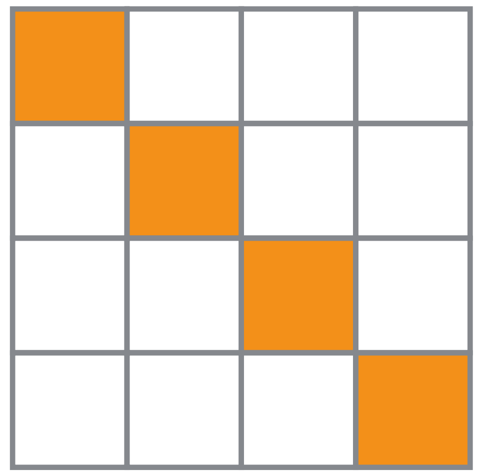


Observing whole arrays, with copying, mapping, and transposing

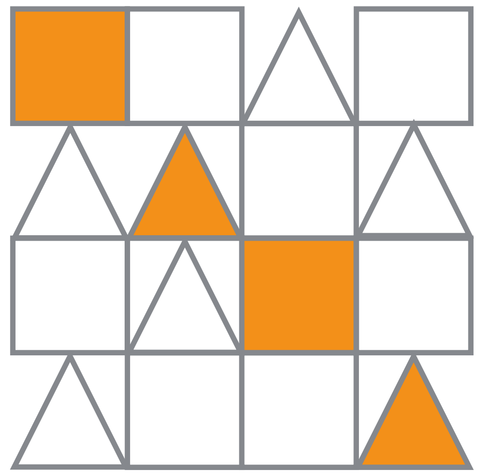


Models from R2 probabilistic programming suite

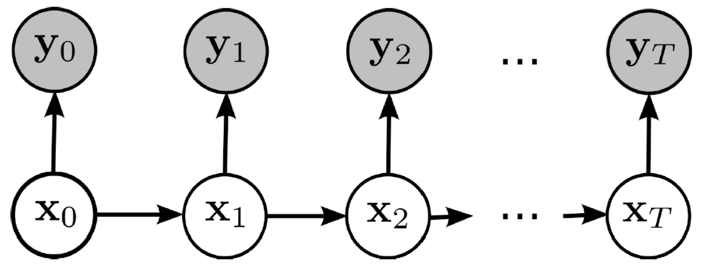
What we want to work



Observing parts of arrays

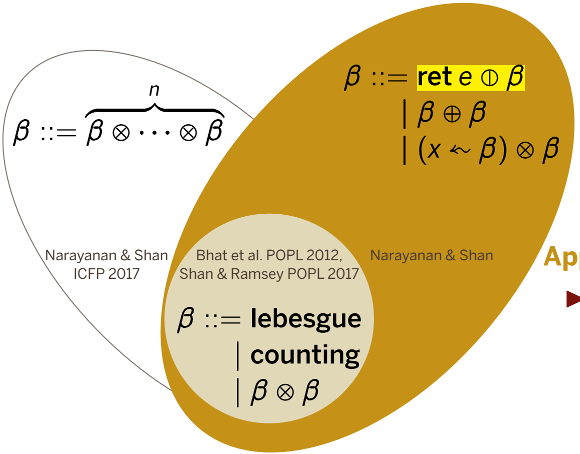


Conditionals inside arrays



Chain-like dependencies between array elements

A growing variety of base measures



Discrete-continuous mixtures
Disjoint sums
Dependent products

Applications:

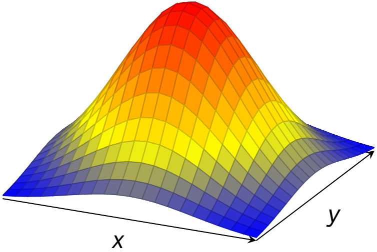
- clamped model/observation (GPA, Tobit, camera)
  - likelihood ratio, importance sampling, mutual information
  - belief update, Gibbs sampling
- Metropolis-Hastings sampling
  - single site
  - reversible jump, light transport

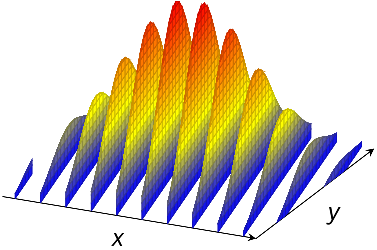
Clamped observation requires mixed base

$x \leftsquigarrow$ **normal**
$y \leftsquigarrow$ **normal**
$t = 5 \cdot x + 0.1 \cdot y$
$t' = \max\{-3, \min\{+3, t\}\}$
$p = (x, y)$

$\neq$

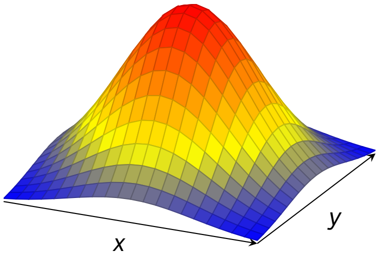
$t' \leftsquigarrow$ **lebesgue**
$p \leftsquigarrow$ **???**

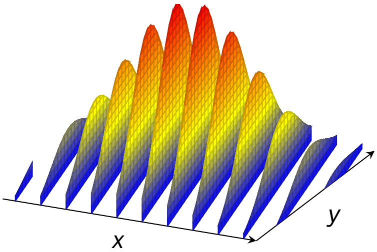




Clamped observation requires mixed base

$$
\begin{aligned}
x &\leftsquigarrow \textbf{normal} \\
y &\leftsquigarrow \textbf{normal} \\
t &= 5 \cdot x + 0.1 \cdot y \\
t' &\leftsquigarrow (\textbf{factor}\ \langle t < -3 \rangle ;\ \textbf{ret}\ {-3}) \oplus \\
&\quad\ (\textbf{factor}\ \langle t < +3 \rangle ;\ \textbf{ret}\ {+3}) \oplus \\
&\quad\ (\textbf{factor}\ \langle |t| \leq 3 \rangle ;\ \textbf{ret}\ t\ ) \\
p &= (x, y)
\end{aligned}
\quad = \quad
\begin{aligned}
t' &\leftsquigarrow \textbf{ret}\ {-3} \oplus \textbf{ret}\ {+3} \oplus \textbf{lebesgue} \\
p &\leftsquigarrow \textbf{???}
\end{aligned}
$$





Base-checking disintegration

$x \leftsquigarrow$ **normal**
$y \leftsquigarrow$ **normal**
$t = 5 \cdot x + 0.1 \cdot y$
factor $\langle t < -3 \rangle$
$t' = -3$
$p = (x, y)$

$=$

$t' \leftsquigarrow$ **ret** $-3 \oplus$ **ret** $+3 \oplus$ **lebesgue**
$p \leftsquigarrow$ **???**

$x \leftsquigarrow$ **normal**
$y \leftsquigarrow$ **normal**
$t = 5 \cdot x + 0.1 \cdot y$
factor $\langle |t| \leq 3 \rangle$
$p = (x, y)$

$=$

$t \leftsquigarrow$ **ret** $-3 \oplus$ **ret** $+3 \oplus$ **lebesgue**
$p \leftsquigarrow$ **???**

Base-checking disintegration

$x \leftsquigarrow$ **normal**
$y \leftsquigarrow$ **normal**
$t = 5 \cdot x + 0.1 \cdot y$
factor $\langle t < -3 \rangle$
$t' = -3$
$p = (x, y)$

density

$t' \leftsquigarrow$ **ret** $-3 \oplus$ **ret** $+3 \oplus$ **lebesgue**
factor $\langle t' = -3 \rangle$
$x \leftsquigarrow$ **normal**
$y \leftsquigarrow$ **normal**
$t = 5 \cdot x + 0.1 \cdot y$
factor $\langle t < -3 \rangle$
$p = (x, y)$

$x \leftsquigarrow$ **normal**
$y \leftsquigarrow$ **normal**
$t = 5 \cdot x + 0.1 \cdot y$
factor $\langle |t| \leq 3 \rangle$
$p = (x, y)$

density

$x \leftsquigarrow$ **normal**
$y \leftsquigarrow$ **ret** $(10 \cdot (-3 - 5 \cdot x)) \oplus$
ret $(10 \cdot (+3 - 5 \cdot x)) \oplus$
lebesgue
$t = 5 \cdot x + 0.1 \cdot y$
factor $\langle t \notin \{\pm 3\} \rangle$
factor $($**dnorm** $y)$
factor $\langle |t| \leq 3 \rangle$
$p = (x, y)$

reparameterize

$t \leftsquigarrow$ **ret** $-3 \oplus$ **ret** $+3 \oplus$ **lebesgue**
$x \leftsquigarrow$ **normal**
$y = 10 \cdot (t - 5 \cdot x)$
factor 10
factor $\langle t \notin \{\pm 3\} \rangle$
factor $($**dnorm** $y)$
factor $\langle |t| \leq 3 \rangle$
$p = (x, y)$

Base-inferring disintegration

$x \leftsquigarrow \textbf{normal}$ $y \leftsquigarrow \textbf{normal}$ $t = 5 \cdot x + 0.1 \cdot y$ $\textbf{factor}\ \langle t < -3 \rangle$ $t' = -3$ $p = (x, y)$	$=$	$t' \leftsquigarrow \mathbb{B}$ $p \leftsquigarrow \textbf{???}$
$x \leftsquigarrow \textbf{normal}$ $y \leftsquigarrow \textbf{normal}$ $t = 5 \cdot x + 0.1 \cdot y$ $\textbf{factor}\ \langle \lvert t \rvert \leq 3 \rangle$ $p = (x, y)$	$=$	$t \leftsquigarrow \mathbb{B}$ $p \leftsquigarrow \textbf{???}$

Base-inferring disintegration

$x \leftsquigarrow$ **normal**
$y \leftsquigarrow$ **normal**
$t = 5 \cdot x + 0.1 \cdot y$
factor $\langle t < -3 \rangle$
$t' = -3$
$p = (x, y)$

density

$t' \leftsquigarrow \mathrm{B}$
divide (**ret** -3) B t'
$x \leftsquigarrow$ **normal**
$y \leftsquigarrow$ **normal**
$t = 5 \cdot x + 0.1 \cdot y$
factor $\langle t < -3 \rangle$
$p = (x, y)$

$x \leftsquigarrow$ **normal**
$y \leftsquigarrow$ **normal**
$t = 5 \cdot x + 0.1 \cdot y$
factor $\langle |t| \leq 3 \rangle$
$p = (x, y)$

density

$x \leftsquigarrow$ **normal**
$y \leftsquigarrow$ **reparam** $(y \mapsto 5 \cdot x + 0.1 \cdot y)$ B
$t = 5 \cdot x + 0.1 \cdot y$
factor (10 / **jacobian** $(y \mapsto \cdots)$ B t)
divide lebesgue B t
factor (**dnorm** y)
factor $\langle |t| \leq 3 \rangle$
$p = (x, y)$

reparameterize

$t \leftsquigarrow \mathrm{B}$
$x \leftsquigarrow$ **normal**
$y = 10 \cdot (t - 5 \cdot x)$
factor 10
divide lebesgue B t
factor (**dnorm** y)
factor $\langle |t| \leq 3 \rangle$
$p = (x, y)$

Base-inferring disintegration

$x \leftsquigarrow$ **normal**
$y \leftsquigarrow$ **normal**
$t = 5 \cdot x + 0.1 \cdot y$
factor $\langle t < -3 \rangle$
$t' = -3$
$p = (x, y)$

density

$t' \leftsquigarrow \mathtt{B}$
divide (**ret** -3) $\mathtt{B}\ t'$
$x \leftsquigarrow$ **normal**
$y \leftsquigarrow$ **normal**
$t = 5 \cdot x + 0.1 \cdot y$
factor $\langle t < -3 \rangle$
$p = (x, y)$

$x \leftsquigarrow$ **normal**
$y \leftsquigarrow$ **normal**
$t = 5 \cdot x + 0.1 \cdot y$
factor $\langle |t| \leq 3 \rangle$
$p = (x, y)$

density

$x \leftsquigarrow$ **normal**
$y \leftsquigarrow$ **reparam** $(y \mapsto 5 \cdot x + 0.1 \cdot y)\ \mathtt{B}$
$t = 5 \cdot x + 0.1 \cdot y$
factor (10 / **jacobian** $(y \mapsto \cdots)\ \mathtt{B}\ t$)
divide lebesgue $\mathtt{B}\ t$
factor (**dnorm** y)
factor $\langle |t| \leq 3 \rangle$
$p = (x, y)$

reparameterize

$t \leftsquigarrow \mathtt{B}$
$x \leftsquigarrow$ **normal**
$y = 10 \cdot (t - 5 \cdot x)$
factor 10
divide lebesgue $\mathtt{B}\ t$
factor (**dnorm** y)
factor $\langle |t| \leq 3 \rangle$
$p = (x, y)$

Base-inferring disintegration

Infer principal base measure

$$\textbf{divide}\begin{pmatrix}\textbf{ret}\ -3 & \oplus\\ \textbf{ret}\ +3 & \oplus\\ \textbf{lebesgue} & \end{pmatrix}\ \mathtt{B} \quad \begin{cases}\textbf{divide}\ (\textbf{ret}\ -3)\ \mathtt{B}\ t' \\ \textbf{divide}\ (\textbf{ret}\ +3)\ \mathtt{B}\ t' \\ \textbf{divide lebesgue}\ \mathtt{B}\ t\end{cases}$$

Generalized density calculation

$$\begin{aligned}
&\mathit{density} : \mathbb{M}\,\alpha \to \mathbb{M}\,\alpha \to \alpha \to \{\mathbb{R}_+\} \\
&\mathit{density}\ \mu\ \nu\ t = \text{let}\ \mu' = \mu \otimes \mathbf{return}\ () \\
&\qquad\qquad\qquad\qquad\ \nu' = \nu \otimes \mathbf{return}\ () \\
&\qquad\qquad\qquad\qquad\ \lambda\ = \mathit{infer}\ \mu'\ t + \mathit{infer}\ \nu'\ t \\
&\qquad\qquad\qquad\quad \text{in}\ |\mathit{check}\ \mu'\ \lambda\ t| \div |\mathit{check}\ \nu'\ \lambda\ t|
\end{aligned}$$

Generalized MCMC sampling

$$
\begin{aligned}
&\mathit{mhg} : \mathbb{M}\,\alpha \to (\alpha \to \mathbb{M}\,\alpha) \to \alpha \to \mathbb{M}\,\alpha \\
&\mathit{mhg}\;\eta\;Q\;x = \mathbf{do}\;\{y \leftsquigarrow Q\;x; \\
&\qquad r_{xy} \leftsquigarrow \mathbf{return}\;(\mathit{greenRatio}\;\eta\;Q\;(x,y)); \\
&\qquad a_{xy} \leftsquigarrow \mathbf{return}\;\min(1, r_{xy}); \\
&\qquad b \leftsquigarrow \mathbf{bern}\;a_{xy}; \\
&\qquad \mathbf{return}\;(\mathbf{if}\;b\;\mathbf{then}\;y\;\mathbf{else}\;x)\}
\end{aligned}
$$

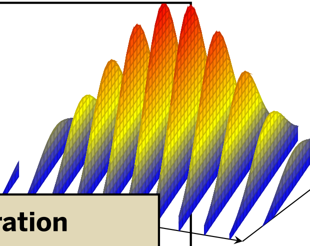
$$
\begin{aligned}
&\mathit{greenRatio} : \mathbb{M}\,\alpha \to (\alpha \to \mathbb{M}\,\alpha) \to (\alpha \times \alpha) \to \{\mathbb{R}_+\} \\
&\mathit{greenRatio}\;\eta\;Q = \mathit{density}\;(\mathit{fmap}\;\mathit{switch}\;(\eta \otimes\!\!= Q))\;(\eta \otimes\!\!= Q)
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{switch} : (\alpha \times \beta) \to (\beta \times \alpha) \\
&\mathit{switch}\;p = (\mathrm{snd}\;p, \mathrm{fst}\;p)
\end{aligned}
$$

$$
\begin{aligned}
&(\otimes\!\!=) : \mathbb{M}\,\alpha \to (\alpha \to \mathbb{M}\,\beta) \to \mathbb{M}\,(\alpha \times \beta) \\
&m \otimes\!\!= k = \mathbf{do}\;\{x \leftsquigarrow m; y \leftsquigarrow k\;x; \mathbf{return}\;(x,y)\}
\end{aligned}
$$

Exact calculation

for approximate computation



https://github.com/pravnar/disintegrating-mixtures

Automatic differentiation

for gradient descent

Automatic disintegration

for inference and sampling

Generalizes density, conditioning

Derived by equational reasoning

A variety of base measures

Mixtures, disjoint sums,
dependent products

Infer principal base measure